On Problem 57 of the Scottish Book

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Dedicated to the memory of Ákos Császár

Abstract

One of the problems in the Schottish book was solved by a construction of M. Laczkovich. In this note we give a Baire-category proof.

Problem No. 57 of the Schottish book reads as follows.

(Ruziewicz) *Given two functions $w(h)$ and $\varphi(h)$, decreasing with $|h|$ to 0, and satisfying the conditions*

$$\lim_{h\to 0}\frac{w(h)}{|h|}=\infty$$

and

$$\lim_{h\to 0}\frac{w(h)}{\varphi(h)}=\infty.$$

Does there exist a function satisfying

$$|f(x+h)-f(x)|<w(h) \tag{1}$$

and

$$\limsup_{h\to 0}\left|\frac{f(x+h)-f(x)}{\varphi(h)}\right|=\infty? \tag{2}$$

(1) and (2) must be satisfied for all x and $h\neq 0$. The affirmative answer was given by Miklós Laczkovich via a construction, see [1, pp. 132–137]. In this note we give a Baire-category proof.

For $x\geq 0$ we set $W(x)=\min(w(x),w(-x))$ and $\sigma(x)=\min(\varphi(x),\varphi(-x))$. Then W,σ are increasing, and by assumption $W(x)/x\to\infty$ and $W(x)/\sigma(x)\to\infty$ as $x\to 0+0$.

Let ω be the largest subadditive (i.e. with the property $\omega(t+s)\leq\omega(t)+\omega(s)$) increasing function that is $\leq W$. Since the supremum of subadditive functions is subadditive, ω exists. There is a sequence $t_m\searrow 0$ for which $\omega(t_m)=W(t_m)$.

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Indeed, t_m can be the smallest $x > 0$ for which $W(x) \le mx$ (this smallest x exists since W is increasing and $W(x)/x \to \infty$ as $x \to 0+0$, furthermore $W(T_m) = mt_m$ is also satisfied), because then the function $g(u) = mu$ if $0 \le u \le t_m$ and $g(u) = W(t_m)$ if $u > t_m$, is subadditive and $g(t_m) = W(t_m)$. But then $\omega(t_m) = W(t_m)$ is also true.

Set

$$T_m = W(t_m)/\sigma(t_m).$$

As we have just mentioned, this tends to ∞ as $m \to \infty$.

We shall work with 1-periodic continuous functions on the real line, and for them we define the "norm"

$$\|f\| = |f(0)| + \sup_{x,h,\ h\neq 0} \frac{|f(x+h) - f(x)|}{\omega(|h|)}.$$

Let B be the closure in this norm of the set of 1-periodic piecewise linear continuous functions (the norm is finite for every such piecewise linear function). We shall use the category theorem in B.

For an $L > 0$ let F_L be the set of those functions f in B for which there is an x such that $|f(x+t) - f(x)| \le L\varphi(t)$ for all t. We will show that F_L is nowhere dense in B. Then any $f \in B \setminus \cup_n F_n$ with $\|f\| < 1/2$ satisfies (2) for all x and

$$|f(x+h) - f(x)| \le \frac{1}{2}\omega(|h|) \le \frac{1}{2}W(|h|) \le \frac{1}{2}w(h) < w(h),$$

so f gives a positive answer to problem No. 57.

Thus, it is left to prove that each F_L is nowhere dense in B. Let O be an arbitrary open subset of B. Then there are a 1-periodic piecewise linear continuous $f_0 \in O$ and an $\varepsilon > 0$, such that the 12ε-neighborhood of f_0 is a subset of O. The graph of f_0 over $[0,1]$ consists of finitely many segments I of the form $\{(u, a+bu) \mid u \in [c,d]\}$. Let us call $[c,d]$ the base of I. Let t_m be so small that each such base is of length $\ge t_m$. Then the graph of f_0 can be considered consisting of segments with base of length in between t_m and $2t_m$ (just decompose each base into intervals of length in between t_m and $2t_m$). Now if the slope of such a segment I of the graph is in absolute value $\ge \varepsilon\omega(t_m)/t_m$, then keep that segment, otherwise replace I by two (connected) segments of slope $\pm 4\varepsilon\omega(t_m)/t_m$ that connect the endpoints of I. There are two possibilities for constructing these new segments—it does not matter which one we choose. This way we get the graph of a function f_1, which we consider to be extended to the whole real line as a 1-periodic function.

Clearly, f_1 is continuous and piecewise linear, so it is in B. It is easy to see that $|f_0 - f_1| \le 4\varepsilon\omega(t_m)$, and $|(f_0 - f_1)'| \le 5\varepsilon\omega(t_m)/t_m$ (where the derivative exists) and the graph of f_1 consists of segments with base of length in between $t_m/4$ and $2t_m$. The subbadditivity of ω implies that $\omega(t_m) \le (1 + t_m/h)\omega(h)$ for $h \le t_m$, and these properties immediately yield that $\|f_0 - f_1\| \le 10\varepsilon$, so f_1 belongs to O.

On the other hand, if $x \in [0,1]$ (by 1-periodicity it is sufficient to consider only points in $[0,1]$), then the point $(x, f_1(x))$ belongs to a segment I of the graph of f_1 with base $[c,d]$ (where, as we have just mentioned, $t_m/4 \le |d-c| \le 2t_m$). Now for $x \le d - t_m/8$ the graph of f_1 over $[x, x+t_m/8]$ is part of the same I, therefore, in view of $|f_1'| \ge \varepsilon\omega(t_m)/t_m$, we have $|f_1(x+t_m/8) - f_1(x)| \ge \varepsilon\omega(t_m)/8$. In the opposite case, i.e. when $d - t_m/8 < x \le d$, we have two possibilities: either $|f_1(x+t_m/8) - f_1(x)| \ge \varepsilon\omega(t_m)/16$, or $|f_1(x+t_m/8) - f_1(x)| < \varepsilon\omega(t_m)/16$. The latter implies (in view of $|f_1'| \ge \varepsilon\omega(t_m)/t_m$ and the fact that now the points $x+t_m/8, x+t_m/4$ belong to the base adjacent to (c,d))

$$\begin{aligned}|f_1(x+t_m/4) - f_1(x)| &\ge |f_1(x+t_m/4) - f_1(x+t_m/8)| - |f_1(x+t_m/8) - f_1(x)| \\ &\ge \varepsilon\omega(t_m)/8 - \varepsilon\omega(t_m)/16 = \varepsilon\omega(t_m)/16.\end{aligned}$$

Hence, in any case at least for one of the choices $h = t_m/8, t_m/4$ we have

$$|f_1(x+h) - f_1(x)| \ge \frac{\varepsilon}{16}\omega(t_m) = \frac{\varepsilon}{16}W(t_m) = \frac{\varepsilon}{16}T_m\sigma(t_m). \tag{3}$$

The same argument shows that for any $x \in [0,1]$ at least for one of the choices $h = t_m/8, t_m/4$ we have

$$|f_1(x-h) - f_1(x)| \ge \frac{\varepsilon}{16}T_m\sigma(t_m). \tag{4}$$

But $\sigma(t_m) = \varphi(t_m)$ or $\sigma(t_m) = \varphi(-t_m)$, so along with (3) and (4) we also have for at least one of the choices $h = h_x = \pm t_m/8, \pm t_m/4$ the inequlity

$$|f_1(x+h) - f_1(x)| \ge \frac{\varepsilon}{16}T_m\sigma(t_m) \ge \frac{\varepsilon}{16}T_m\varphi(h). \tag{5}$$

But then in a small neighborhood $O_m \subset O$ of f_1, which may depend on m but not on $x \in [0,1]$, each function f satisfies

$$|f(x+h) - f(x)| \ge \frac{\varepsilon}{32}T_m\varphi(h). \tag{6}$$

for the same value $h = h_x = \pm t_m/8$ or $\pm t_m/4$.

So far t_m was sufficiently small. Let it now be so small that $\varepsilon T_m > 32L$ is true. Then, in view of (6), O_m is disjoint from F_L, which shows that F_L is, indeed, nowhere dense in B.

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Remarks. 1. Laczkovich proved more, namely there is an f with property (2) and either with

$$\limsup_{h\to 0+0} \left|\frac{f(x+h) - f(x)}{\varphi(h)}\right| = \infty \qquad \text{for all } x \tag{7}$$

or with

$$\limsup_{h \to 0-0} \left| \frac{f(x+h) - f(x)}{\varphi(h)} \right| = \infty \qquad \text{for all } x \tag{8}$$

(but not necessarily both!). The proof above gives precisely the same (namely (7) is true if $\sigma(t_m) = \varphi(t_m)$ for infinitely many m, and (8) is true if $\sigma(t_m) = \varphi(-t_m)$ for infinitely many m).

2. As to the origin of the problem, it is clearly an extension of the fact that continuous functions need not be differentiable. For example, the choice $\varphi(t) = t$ and $w(t) = t^\alpha$, $0 < \alpha < 1$, yields the existence of nowhere differentiable Lip α functions.

References

[1] R. D. Mauldin, *The Scottish Book*, second edition, Birhäuser, Springer International Publishing AG, Cham, Heidelberg, New York, Dodrecht, London, 2015.

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